Introduction to Cluster Algebras

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Overview

- Cluster algebras are commutative rings with distinguished generators (cluster variables) having a remarkable combinatorial structure.
- The structure of a cluster algebra is encoded by a quiver, and the relations among the cluster variables are encoded by quiver mutation.
- Cluster algebras were introduced by Fomin and Zelevinsky in 2000, motivated by total positivity and Lusztig’s canonical basis.

Cluster algebras have since appeared in many other contexts such as:
- Poisson geometry
- triangulations of surfaces and Teichmüller theory
- tropical geometry
- mathematical physics: wall-crossing phenomena, quiver gauge theories, scattering amplitudes, soliton solutions to the KP equation
Talk 1: What is a cluster algebra?
- Motivation from total positivity
- Quivers and quiver mutation
- Seeds and seed mutation
- Definition of cluster algebra
- Cluster algebras in nature: surfaces, Grassmannians

Talk II: Cluster structures in commutative rings
- How can we identify a commutative ring with a cluster algebra?
- Starfish lemma
- The Grassmannian, revisited
- Presentations by generators and relations?
The Grassmannian and its positive part

The Grassmannian \( Gr_{k,n}(\mathbb{R}) = \{ V \mid V \subset \mathbb{R}^n, \dim V = k \} \)

Represent an element of \( Gr_{k,n}(\mathbb{R}) \) by a full-rank \( k \times n \) matrix \( A \).

\[
\begin{pmatrix}
1 & 0 & -1 & -2 \\
0 & 1 & 3 & 2
\end{pmatrix}
\]

Can think of \( Gr_{k,n}(\mathbb{R}) \) as \( \text{Mat}_{k,n}/\sim \).

Given \( I \in \binom{[n]}{k} \), the Plücker coordinate \( \Delta_I(A) \) is the minor of the \( k \times k \) submatrix of \( A \) in column set \( I \).

The \textit{totally positive part} of the Grassmannian \( (Gr_{k,n})_{>0} \) is the subset of \( Gr_{k,n}(\mathbb{R}) \) where all Plücker coordinates \( \Delta_I(A) > 0 \).

A \( k \times n \) matrix \( A \) has \( \binom{n}{k} \) Plücker coordinates.

How many (and which ones) do we need to test to determine whether \( A \) represents a point of \( (Gr_{k,n})_{>0} \)?
The Grassmannian and its positive part

Represent an element of $Gr_{k,n}(\mathbb{R})$ by a full-rank $k \times n$ matrix $A$.

\[
\begin{pmatrix}
1 & 0 & -1 & -2 \\
0 & 1 & 3 & 2
\end{pmatrix}
\]

Given $I \in \binom{[n]}{k}$, the \textit{Plücker coordinate} $\Delta_I(A)$ is the minor of the $k \times k$ submatrix of $A$ in column set $I$.

The Plücker coordinates satisfy

\[\Delta_{13}(A)\Delta_{24}(A) = \Delta_{12}(A)\Delta_{34}(A) + \Delta_{14}(A)\Delta_{23}(A).\]

So if $\Delta_{12}, \Delta_{23}, \Delta_{34}, \Delta_{14}$ and $\Delta_{24}$ are positive, so is $\Delta_{13}$.
Or if $\Delta_{12}, \Delta_{23}, \Delta_{34}, \Delta_{14}$ and $\Delta_{13}$ are positive, so is $\Delta_{24}$.

How can we generalize this picture to $Gr_{2,n}(\mathbb{R})$? $Gr_{k,n}(\mathbb{R})$?
A *quiver* is a finite directed graph. Multiple edges are allowed. Oriented cycles of length 1 or 2 are forbidden. Two types of vertices: “frozen” and “mutable.” Ignore edges connecting frozen vertices.
Let $k$ be a mutable vertex of $Q$.

**Quiver mutation** $\mu_k : Q \mapsto Q'$ is computed in 3 steps:

1. For each instance of $j \to k \to \ell$, introduce an edge $j \to \ell$.
2. Reverse the direction of all edges incident to $k$.
3. Remove oriented 2-cycles.

**Mutation is an involution,** i.e. $\mu_k^2(Q) = Q$ for each vertex $k$.

**Two quivers are mutation-equivalent** if one can get between them via a sequence of mutations. _Show aplet._
Let $\mathcal{F}$ be a field of rational functions in $m$ independent variables over $\mathbb{C}$. A \textit{seed} in $\mathcal{F}$ is a pair $(Q, x)$ consisting of:

- a quiver $Q$ on $m$ vertices
- an \textit{extended cluster} $x$, an $m$-tuple of algebraically independent (over $\mathbb{C}$) elements of $\mathcal{F}$, indexed by the vertices of $Q$.

\begin{align*}
\text{coefficient variables} & \leftrightarrow \text{frozen vertices} \\
\text{cluster variables} & \leftrightarrow \text{mutable vertices}
\end{align*}

\begin{align*}
\text{Cluster} &= \{\text{cluster variables}\} \\
\text{Extended Cluster} &= \{\text{cluster variables, coefficient variables}\}
\end{align*}
Let $k$ be a mutable vertex in $Q$ and let $x_k$ be the corresponding cluster variable. Then the seed mutation $\mu_k : (Q, x) \mapsto (Q', x')$ is defined by

- $Q' = \mu_k(Q)$
- $x' = x \cup \{x'_k\} \setminus \{x_k\}$, where

$$x_k x'_k = \prod_{j \leftarrow k} x_j + \prod_{j \rightarrow k} x_j$$

(is the exchange relation)

Remark: Mutation is an involution.

Example

\[
\begin{array}{ccc}
\begin{array}{c}
\begin{array}{ccc}
  x_1 & \leftrightarrow & x_2 \\
  \downarrow & & \downarrow \\
  x_4 & \leftrightarrow & x_5 \\
  \downarrow & & \downarrow \\
  x_6 & \leftrightarrow & x_3 \\
\end{array}
\end{array}
& \mu_2 & \begin{array}{c}
\begin{array}{ccc}
  x_1 & \leftrightarrow & x_2 \\
  x_3 & \leftrightarrow & \frac{x_1^2 x_3 + x_5}{x_3} \\
  \downarrow & & \downarrow \\
  x_4 & \leftrightarrow & x_5 \\
  \downarrow & & \downarrow \\
  x_6 & \leftrightarrow & x_3 \\
\end{array}
\end{array}
\end{array}
\]
Let \((Q, x)\) be a seed in \(\mathcal{F}\), where \(Q\) has \(n\) mutable vertices. Consider the \(n\)-regular tree \(\mathbb{T}\) with vertices labeled by seeds, obtained by applying all possible sequences of mutations to \((Q, x)\).

Let \(\chi\) be the union of all cluster variables which appear at nodes of \(\mathbb{T}\).

Let the \textit{ground ring} be \(\mathcal{R} = \mathbb{C}[x_{n+1}, \ldots, x_m]\), the polynomial ring generated by frozen variables. (Alternatively let \(\mathcal{R} = \mathbb{C}[x_{n+1}^\pm, \ldots, x_m^\pm]\).)

The \textit{cluster algebra} \(\mathcal{A}(Q) := \mathcal{R}[\chi] \subset \mathcal{F}\) is the \(\mathcal{R}\)-subalg generated by \(\chi\).
Example

Consider the following seed \((Q, x)\), where \(x = \{x_1, x_2\}\).

![Diagram of a 2-regular tree closing up to form a pentagon.](image)

The cluster algebra \(A(Q)\) is the subring of \(F = \mathbb{C}(x_1, x_2)\) generated by all cluster variables \(\chi = \{x_1, x_2, \frac{1+x_2}{x_1}, \frac{1+x_1+x_2}{x_1x_2}, \frac{1+x_1}{x_2}\}\).

Note: every cluster variable is a Laurent polynomial in \(\{x_1, x_2\}\).
Note: each Laurent polynomial has positive coefficients.
Note: there are finitely many cluster variables.
The 2-regular tree closes up to form a pentagon.
Fundamental results

Let $A = A(Q)$ be an arbitrary cluster algebra, with initial seed $(Q, x)$.

**Laurent phenomenon (Fomin + Zelevinsky)**

Every cluster variable is a Laurent polynomial in the variables from $x$ (the *initial cluster variables*).

**Positivity Theorem (Lee-Schiffler, Gross-Hacking-Keel)**

Each such Laurent polynomial has positive coefficients.

**Finite type classification (Fomin + Zelevinsky)**

We say $A$ has *finite type* if there are only finitely many cluster variables. The finite type cluster algebras are classified by Dynkin diagrams. When $A$ is of finite type, the $n$-regular tree closes up on itself and becomes the 1-skeleton of a convex polytope.
Fix a triangulation $T$ of a $d$-gon. We associate to it a quiver $Q_T$:

This gives rise to a cluster algebra $\mathcal{A}(Q_T)$, with initial seed $(Q_T, \{x_1, \ldots, x_{2d-3}\})$. 

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The set of triangulations of a polygon is connected by \textit{flips}. 
Flips correspond to mutations.

Note that $\mu_2(Q_T) = Q_{T'}$. 
Cluster algebras and triangulations of a polygon

Triangulations are connected by flips, and flips $\leftrightarrow$ mutations. Moreover:

- seeds $\leftrightarrow$ the triangulations of the polygon
- coefficient variables $\leftrightarrow$ the $d$ sides of the polygon
- cluster variables $\leftrightarrow$ the $\frac{d(d - 3)}{2}$ diagonals of the polygon

Exchange relation:

$$x_h x'_h = x_i x_k + x_j x_\ell$$
Cluster algebras and triangulations of a polygon

Relabel the triangulation, and coefficient/cluster variables as follows:

![Triangulation diagram]

Exchange relation:

\[
p_{ad} \quad p_{ac} \quad p_{bc} \quad p_{cd}
\]

\[
p_{ab} \quad p_{bd} = p_{ab}p_{cd} + p_{bc}p_{ad}
\]

This identifies our cluster algebra with the coordinate ring of the Grassmannian \( \mathbb{C}[Gr_{2,d}] \)! Every cluster (triangulation) gives rise to a positivity test for membership in \((Gr_{2,d})_0\). (Why?)
The cluster algebra associated to $\mathbb{C}[Gr_{2,d}]$ can be visualized using the *associahedron*:
Two generalizations of this cluster algebra

Cluster algebra from
triangulations of a polygon

\[ \mathbb{C}[Gr_{2,d}] \]

Cluster algebra from
triangulations of a Riemann surface

Teichmuller theory

The coordinate ring

\[ \mathbb{C}[Gr_{k,d}] \]
First generalization: polygon $\rightsquigarrow$ surface

Recall that given a triangulation of a polygon, we can construct a quiver and an associated cluster algebra.

Idea: if we have an oriented surface with some marked points, we can triangulate it, and construct a quiver as before!
Second generalization: triangulation $\rightsquigarrow$ plabic graph

To get positivity tests for $(Gr_{k,n})_{>0}$ for $k > 2$, replace triangulations with Postnikov's $(k,n)$-plabic graphs.
To get polytope encoding these positivity tests for $(Gr_{k,n})_{>0}$, replace associahedron with higher associahedra (Galashin-Postnikov-W.)
WHERE YOU SIT IN CLASS/SEMINAR
And what it says about you:

Mid-Center: “Bring it on.”

Nearest Exit: Uncommitted

Back Row: “Too cool for school”

Front Row: Teacher’s pet wannabes

Second-row sleepers: Good intentions, bad narcolepsy

Against the wall: “I’m sensitive. Please ignore me.”

Proximity to Lecturer:

X = \frac{How much you care}{How sleepy you are}
References